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EIGENVALUES AND EIGENVECTORS OF CORRELATION MATRICES

WITH SPECIAL STRUCTURES: APPLICATIONS TO FACTOR

ANALYSIS OF COMMON STOCK RATES OF RETURN

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Abstract:

General forms for the eigenvalues and eigenvectors of certain patterned

correlation matrices are obtained. The results are discussed in the context of

a factor analysis of common stock rates of return.

1. Introduction

We derive the general forms for the eigenvalues and eigenvectors of certain correlation matrices with special structures. Our inquiry was motivated by an analysis of the rates of return on common stocks. Stocks within certain groups (for example, stocks within the same industry) may be equally correlated and the correlations between stocks in different groups, although equal, may be smaller in magnitude than the within group correlations.

The correlation matrices we examine are composed of blocks of equally cor-related groups. The correlations within each block are expressed as simple powers (0,1,2 and 3) of a basic correlation ρ. For example, with

$$A = \begin{bmatrix} 1 & \rho & . & . & . & \rho \\ \rho & 1 & . & . & . & \rho \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ \rho & \rho & . & . & . & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \rho^2 & \rho^2 & . & . & . & \rho^2 \\ \rho^2 & \rho^2 & . & . & . & \rho^2 \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ \rho^2 & \rho^2 & . & . & . & \rho^2 \end{bmatrix}$$

A is (mxm) and B is (mxn)

we can form the "two group" correlation matrix:

$$\underset{(m+n)\times(m+n)}{\underset{\sim}{\rho}} = \begin{bmatrix} A_1 & B \\ \hline B' & A_2 \end{bmatrix}$$

where A_1 and A_2 are analogous to A with dimension (mxm) and (nxn) respectively.

In the following sections we consider several patterned correlation matrices of the type discussed above. Our eigenvalue-eigenvector results can be generally applied to many problems involving a principal component or factor analysis. However, it will be convenient to refer to the groups of variables as groups of common stock rates of return and to interpret our work in terms of the

factor structure for stock prices embedded in the Arbitrage Pricing Theory (APT) of assets proposed by Ross (1976). We provide a small example of factor structure with five stocks from two industries.

Specifically, we consider correlation matrices corresponding to the following cases:

1. Two groups of equally correlated variables (stocks) with m=n (equal group sizes) and m≠n (unequal group sizes)

2. Three groups of equally correlated variables (stocks) with m=n=p (equal group sizes)

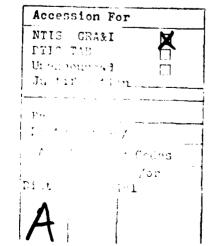3. Four groups of equally correlated variables with m=n=p=q (equal group sizes)

2. Patterned Correlation Matrices for Two Equally Correlated Groups

We define

$$
A = \begin{bmatrix} 1 & \rho & \cdot & \cdot & \cdot & \rho \\ \rho & 1 & \cdot & \cdot & \cdot & \rho \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ \rho & \rho & \cdot & \cdot & \cdot & 1 \end{bmatrix} \quad ; \qquad B = \begin{bmatrix} \rho^2 & \rho^2 & \cdot & \cdot & \cdot & \rho^2 \\ \rho^2 & \rho^2 & \cdot & \cdot & \cdot & \rho^2 \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ \rho^2 & \rho^2 & \cdot & \cdot & \cdot & \rho^2 \end{bmatrix} \qquad (1)
$$

with $-1 \leq \rho \leq 1$. Our analysis is restricted to $\rho > 0$ and all of our examples deal with this case. The matrix A is square with dimensions determined by the problem. In this section the dimensions of A are (mxm) or (nxn). The matrix B is, in general, rectangular. Here B has dimension (mxn).

Groups of equal sizes: m=n

Using (1), consider the (2mx2m) correlation matrix

$$\rho = \begin{bmatrix} A & B \\ (m \times m) & (m \times m) \\ \hline B & A \\ (m \times m) & (m \times m) \end{bmatrix} \qquad (2)$$

The matrix ρ implies that stocks within the two individual groups are equally

correlated with correlation coefficient ρ. The correlation between any two

stocks from different groups is ρ^2.

Making use of the chosen pattern for ρ, it can be shown that ρ has three

distinct eigenvalues:

$$\lambda^{(1)} = 1 - \rho \text{ (multiplicity } 2(m-1))$$

$$\lambda^{(2)} = 1 + (m-1)\ \rho - m\rho^2 \qquad (3)$$

$$\lambda^{(3)} = 1 + (m-1)\ \rho + m\rho^2$$

with $\lambda^{(3)} > \lambda^{(2)} > \lambda^{(1)} \geq 0$ *(since by assumption $\rho > 0$)*. The $2(m-1)$ eigenvec-

tors (unnormalized) corresponding to the smallest eigenvalue $\lambda^{(1)} = 1 - \rho$ are:

$$\underset{\sim}{V}_1 \quad = [\overbrace{1, -1, 0, \ldots, 0}^{m}; \overbrace{0, \ldots, 0}^{m}]'$$

$$\underset{\sim}{V}_2 \quad = [1, 1, -2, \ldots, 0; 0, \ldots, 0]'$$

$$\vdots \qquad\qquad \vdots$$

$$\underset{\sim}{V}_{m-1} \quad = [1, 1, 1, \ldots, -(m-1); 0, \ldots, 0]' \qquad (4)$$

$$\underset{\sim}{V}_m \quad = [0, \ldots, 0; 1, -1, 0, \ldots, 0]'$$

$$\vdots \qquad\qquad \vdots$$

$$\underset{\sim}{V}_{2(m-1)} = [0, \ldots, 0; 1, 1, 1, \ldots, -(m-1)]'$$

The eigenvectors (unnormalized) corresponding to the remaining eigenvalues are:

For $\lambda^{(2)} = 1 + (m-1)\rho - m\rho^2$,

$$\underset{\sim}{V}_{2m-1} = \begin{bmatrix} 1, & 1, & \ldots, & 1; & -1, & -1, & \ldots, & -1 \end{bmatrix}' \qquad (5)$$

For $\lambda^{(3)} = 1 + (m-1)\rho + m\rho^2$,

$$\underset{\sim}{V}_{2m} = \begin{bmatrix} 1, & 1, & \ldots, & 1 \end{bmatrix}'$$

If we regard the eigenvectors as (unscaled) coefficient vectors for the principal components obtained from the correlation matrix $\underset{\sim}{\rho}$, then we see that the eigenvectors corresponding to the two largest eigenvalues represent a "sum" of all the stocks (variables) and a "contrast" between the stocks (variables) in the two groups respectively. A similar interpretation of the eigenvectors can be given in terms of an underlying factor model (see section 5).

Group of unequal sizes: $m \neq n$

Consider the correlation matrix $\underset{\sim}{\rho}$ in (2) with B of dimension m×n, $m \neq n$ and the second diagonal A matrix of dimension (n×n). It is straightforward to verify that one of the distinct eigenvalues of $\underset{\sim}{\rho}$ is $\lambda^{(1)} = 1 - \rho$ with multiplicity $(m-1) + (n-1) = m+n-2$. The $m + n - 2$ (unnormalized) eigenvectors corresponding to $\lambda^{(1)} = 1 - \rho$ are:

$$\underset{\sim}{V}_1 \quad = \quad \left[\overbrace{1, \ -1, \ 0, \ \dots, \ 0}^{m}; \ \overbrace{0, \ \dots, \ 0}^{n}\right]'$$

$$\underset{\sim}{V}_2 \quad = \quad \left[1, \ 1, \ -2, \ \dots, \ 0; \ 0, \ \dots, \ 0\right]'$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\underset{\sim}{V}_{m-1} = \quad \left[1, \ 1, \ 1, \ \dots, \ -(m-1); \ 0, \ \dots, \ 0\right]' \tag{6}$$

$$\underset{\sim}{V}_m \quad = \quad \left[0, \ \dots, \ 0; \ 1, \ -1, \ 0, \ \dots, \ 0\right]'$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\underset{\sim}{V}_{m+n-2} = \quad \left[0, \ \dots, \ 0; \ 1, \ 1, \ 1, \ \dots, \ -(n-1)\right]'$$

The remaining two eigenvectors and associated distinct eigenvalues can be determined from orthogonality requirements. Specifically, the remaining eigenvectors must be of the form $\underset{\sim}{V} = \left[a, \ a, \ \dots, \ a; \ b, \ b, \ \dots, \ b\right]'$ with both $a \neq 0$ and $b \neq 0$. From the definition of eigenvalues and eigenvectors we thus arrive at two distinct equations:

$$a(1 + (m-1)\rho) + b(n\rho^2) = \lambda a$$

$$a(m\rho^2) + b(1 + (n-1)\rho) = \lambda b$$

We set $a = 1$ without loss of generality so that

$$1 + (m-1)\rho + bn\rho^2 = \lambda$$

$$\frac{m\rho^2}{b} + 1 + (n-1)\rho = \lambda$$

or, equivalently,

$$1 + (m-1)\rho + bn\rho^2 = 1 + (n-1)\rho + \frac{m\rho^2}{b} \tag{7}$$

Solving the quadratic equation (7) for b we obtain:

$$b = \frac{-(m-n)\rho \overset{+}{-} \rho\sqrt{(m-n)^2 + 4mn\rho^2}}{2n\rho^2} = \frac{-(m-n) \overset{+}{-} \sqrt{(m-n)^2 + 4mn\rho^2}}{2n\rho} \qquad (8)$$

Denoting the first (largest) solution for b in (8) by b^+ and the second (smallest) solution for b by b^-, we obtain the remaining two distinct eigenvalues and corresponding (unnormalized) eigenvectors. Thus

$$\lambda^{(2)} = 1 + (m-1)\rho + b^-(n\rho^2)$$

$$\underset{\sim}{V}_{m+n-1} = \left[1, 1, \ldots, 1; b^-, b^-, \ldots, b^-\right]'$$

$$\lambda^{(3)} = 1 + (m-1)\rho + b^+(n\rho^2) \qquad (9)$$

$$\underset{\sim}{V}_{m+n} = \left[1, 1, \ldots, 1; b^+, b^+, \ldots, b^+\right]'$$

Noting that $b^+b^- = -\frac{m}{n}$, it is clear that $\underset{\sim}{V}_{m+n-1}$ and $\underset{\sim}{V}_{m+n}$ are orthogonal and each of these vectors is orthogonal to the eigenvectors $\underset{\sim}{V}_1, \ldots, \underset{\sim}{V}_{m+n-2}$. More-over, since $b^+ > 0$ and $b^+ > b^-$, $\lambda^{(3)} > \lambda^{(2)}$. It can also be shown that $\lambda^{(2)} - \lambda^{(1)} > 0$ and this difference increases as m and/or n increases. Thus we have $\lambda^{(3)} > \lambda^{(2)} > \lambda^{(1)}$ and as m and/or n increases the "importance" of the principal components constructed from the eigenvectors corresponding to $\lambda^{(3)}$ and $\lambda^{(2)}$ increases.

We note that the elements of $\underset{\sim}{V}_{m+n}$, the eigenvector associated with the largest eigenvalue, are all positive and become equal as $\rho \to 1$. The first m elements of the eigenvector $\underset{\sim}{V}_{m+n-1}$ are of different sign than the last n elements. In addition, the sum of the elements of $\underset{\sim}{V}_{m+n-1}$ approaches zero as $\rho \to 1$. Again, from a principal components viewpoint, if ρ is positive and large, the eigenvectors corresponding to the two largest eigenvalues represent a "sum" and a "contrast" of the original m+n variables.

Numerical example

Let $m=3$, $n=2$ and $\rho=.6$ then

$$
\rho_{\sim} = \left[\begin{array}{c|c} \begin{matrix} A_1 \\ (3\times3) \end{matrix} & \begin{matrix} B \\ (3\times2) \end{matrix} \\ \hline \begin{matrix} B' \\ (2\times3) \end{matrix} & \begin{matrix} A_2 \\ (2\times2) \end{matrix} \end{array}\right] = \left[\begin{array}{ccc|cc} 1.00 & 0.60 & 0.60 & 0.36 & 0.36 \\ 0.60 & 1.00 & 0.60 & 0.36 & 0.36 \\ 0.60 & 0.60 & 1.00 & 0.36 & 0.36 \\ \hline 0.36 & 0.36 & 0.36 & 1.00 & 0.60 \\ 0.36 & 0.36 & 0.36 & 0.60 & 1.00 \end{array}\right]
$$

From (8), $b^+ = \dfrac{-1 + \sqrt{1 + 4(6)(0.36)}}{2(2)(0.6)} = 0.8770$

$$b^- = \frac{-1 - \sqrt{1 + 4(6)(0.36)}}{2(2)(0.6)} = -1.7103$$

Thus $\lambda^{(1)} = 1 - \rho = 0.4$ (multiplicity 3), and $\lambda^{(2)}$, $\lambda^{(3)}$ can be determined using $b^+ = 0.8770$, $b^- = -1.7103$ and (9). We have:

Eigenvalues (Ordered according to decreasing magnitudes)

2.8315	0.9685	0.4000	0.4000	0.4000

Normalized eigenvectors (columns)

0.4694	0.3361	0.7071	0.4082	0.0000
0.4694	0.3361	-0.7071	0.4082	0.0000
0.4694	0.3361	0.0000	-0.8165	0.0000
0.4117	-0.5749	0.0000	0.0000	0.7071
0.4117	-0.5749	0.0000	0.0000	-0.7071

3. Correlation Matrix for Three Equally Correlated Groups

We consider three groups of stocks of the same sizes, so $m=n=p$. Using the definitions in (1), consider the $(3m\times3m)$ correlation matrix

$$
\rho_{\sim} = \left[\begin{array}{c|c|c} A & B & B \\ \hline B & A & B \\ \hline B & B & A \end{array}\right] \tag{10}
$$

where both A and B are of dimensions (mxm). Thus stocks within each of the three groups are equally correlated with correlation coefficient ρ. The correlation between any two stocks with each stock from a different group is ρ^2.

For the matrix ρ in (10) there are three distinct eigenvalues. The smallest eigenvalue $\lambda^{(1)} = 1 - \rho$ has multiplicity $3(m-1)$ and the associated (unnormalized) eigenvectors can be written as

$$
\underset{\sim}{V_1} = \left[\overbrace{1, -1, 0, \ldots, 0}^{m}; \overbrace{0, \ldots, 0}^{m}; \overbrace{0, \ldots, 0}^{m} \right]'
$$

$$
\underset{\sim}{V_2} = \left[1, 1, -2, \ldots, 0; 0, \ldots, 0; 0, \ldots, 0 \right]'
$$

$$
\vdots
$$

$$
\underset{\sim}{V_{m-1}} = \left[1, 1, 1, \ldots, -(m-1); 0, \ldots, 0; 0, \ldots, 0 \right]'
$$

$$
\underset{\sim}{V_m} = \left[0, \ldots, 0; 1, -1, 0, \ldots, 0; 0, \ldots, 0 \right]'
$$

$$
\vdots \tag{11}
$$

$$
\underset{\sim}{V_{2m-2}} = \left[0, \ldots, 0; 1, 1, 1, \ldots, -(m-1); 0, \ldots, 0 \right]'
$$

$$
\underset{\sim}{V_{2m-1}} = \left[0, \ldots, 0; 0, \ldots, 0; 1, -1, 0, \ldots, 0 \right]'
$$

$$
\vdots
$$

$$
\underset{\sim}{V_{3m-3}} = \left[0, \ldots, 0; 0, \ldots, 0; 1, 1, 1, \ldots, -(m-1) \right]'
$$

Using the orthogonality requirement, the remaining three eigenvectors have the form $\underset{\sim}{V} = \left[a, \ldots, a; b, \ldots, b; c, \ldots, c \right]$. Assume first that $a \neq 0$ (the case of $a=0$ will be discussed later) and hence, without loss of generality, take $a=1$. Solving $\underset{\sim}{\rho} \underset{\sim}{V} = \lambda \underset{\sim}{V}$ with $\underline{V} = \left[1, \ldots, 1; b, \ldots, b; c, \ldots, c \right]$ yields three separate equations

$$1 + (m-1)\rho + m\rho^2 b + m\rho^2 c = \lambda$$

$$m\rho^2 + (1 + (m-1)\rho)b + m\rho^2 c = \lambda b \tag{12}$$

$$m\rho^2 + m\rho^2 b + (1 + (m-1)\rho)c = \lambda c$$

Multiplying the first equation in (12) by b and subtracting the second equation gives

$$(b^2 - 1)m\rho^2 + m\rho^2 c (b - 1) = 0$$

A similar manipulation using the first and third equations in (12) gives

$$(c^2 - 1)m\rho^2 + m\rho^2 b(c - 1) = 0$$

Since $m\rho^2 \neq 0$ (for $\rho \neq 0$), we can write

$$b^2 - 1 + c(b - 1) = 0$$

$$c^2 - 1 + b(c - 1) = 0 \tag{13}$$

or, after subtracting the second equation immediately above from the first,

$$b^2 - c^2 + (b - c) = 0$$

Consequently, $(b - c)(b + c + 1) = 0$ and hence $b = c$ or $b + c + 1 = 0$.

Case I: Suppose $b = c$. Solving the first equation in (13) gives
$b = c = 1$ or $b = c = -1/2$

Case II: Suppose $b + c + 1 = 0$. Any combination of b and c satisfying $b + c + 1 = 0$ solves (13) and consequently there are an infinite number of solutions.

To summarize, with $a = 1$, the condition $b = c = 1$ gives the largest eigen-value, $\lambda^{(3)}$, and corresponding eigenvector

$$\lambda^{(3)} = 1 + (m - 1)\rho + 2m\rho^2$$

$$\underset{\sim}{V}_{3m} = \begin{bmatrix} 1, & \ldots, & 1; & 1, & \ldots, & 1; & 1, & \ldots, & 1 \end{bmatrix}' \tag{14}$$

Using (12) with $b = c = -1/2$ or $b + c + 1 = 0$, we get the second largest eigen-value, $\lambda^{(2)}$, and two associated eigenvectors

$$\lambda^{(2)} = 1 + (m - 1)\rho - m\rho^2$$

$$\underset{\sim}{V}_{3m-2} = \begin{bmatrix} 1, & \ldots, & 1; & -1/2, & \ldots, & -1/2; & -1/2, & \ldots, & -1/2 \end{bmatrix} \tag{15}$$

$$\underset{\sim}{V}_{3m-1} = \begin{bmatrix} 1, & \ldots, & 1; & b, & \ldots, & b; & c, & \ldots, & c \end{bmatrix}'$$

with $b + c + 1 = 0$

Note the vectors in (15) are not orthogonal.

Starting with $b = 1$ or $c = 1$ gives the following:

$\underline{b = 1}$

$$\lambda^{(3)} = 1 + (m-1)\rho + 2m\rho^2$$

$$\underset{\sim}{V}_{3m} = \begin{bmatrix} 1, & \ldots, & 1; & 1, & \ldots, & 1; & 1, & \ldots, & 1 \end{bmatrix}'$$

$$\lambda^{(2)} = 1 + (m-1)\rho - m\rho^2 \tag{16}$$

$$\underset{\sim}{V}_{3m-2} = \begin{bmatrix} -1/2, & \ldots, & -1/2; & 1, & \ldots, & 1; & -1/2, & \ldots, & -1/2 \end{bmatrix}'$$

$$\underset{\sim}{V}_{3m-1} = \begin{bmatrix} a, & \ldots, & a; & 1, & \ldots, & 1; & c, & \ldots, & c \end{bmatrix}'$$

with $a + c + 1 = 0$

c = 1

$$\lambda^{(3)} = 1 + (m-1)\rho + 2m\rho^2$$

$$\underset{\sim}{V}_{3m} = [1, \ldots, 1; 1, \ldots, 1; 1, \ldots, 1]'$$

$$\lambda^{(2)} = 1 + (m-1)\rho - m\rho^2$$

$$\underset{\sim}{V}_{3m-2} = [-1/2, \ldots, -1/2; -1/2, \ldots, -1/2; 1, \ldots, 1]'$$

$$\underset{\sim}{V}_{3m-1} = [a, \ldots, a; b, \ldots, b; 1, \ldots, 1]'$$

with $a + b + 1 = 0$

Combining the results in (15), (16) and (17) we see that some of the eigen-vectors corresponding to the eigenvalue $\lambda^{(2)} = 1 + (m-1)\rho - m\rho^2$ can be written as linear combinations of the other eigenvectors corresponding to this eigen-value. If we exclude the redundant eigenvectors and orthogonalize the remaining two eigenvectors we have the final solution:

$$\lambda^{(3)} = 1 + (m-1)\rho + 2m\rho^2$$

$$\underset{\sim}{V}_{3m} = [1, \ldots, 1; 1, \ldots, 1; 1, \ldots, 1]'$$

$$\lambda^{(2)} = 1 + (m-1)\rho - m\rho^2 \quad \text{(multiplicity 2)} \tag{18}$$

$$\underset{\sim}{V}_{3m-2} = [3/2, \ldots, 3/2; 0, \ldots, 0; -3/2, \ldots, -3/2]'$$

$$\underset{\sim}{V}_{3m-1} = [-1/2, \ldots, -1/2; 1, \ldots, 1; -1/2, \ldots, -1/2]'$$

Recall our discussion leading to (18) started with a=1 (and b=1 and c=1). It is easy to see that an initial choice a=0 (or b=0 or c=0) leads directly to a particular case already considered for b=1 (or c=1 or a=1). Therefore, we do

not have to investigate the choice a=0 (or b=0 or c=0) and (18) does indeed represent the two largest distinct eigenvalues and their eigenvectors.

A numerical example (principal components and factor analysis

Let m=3 and ρ=.7 so the correlation matrix (10) is (9x9). Using (18) and $\lambda^{(1)} = 1 - \rho$, the eigenvalues of $\underset{\sim}{\rho}$ are:

$$\lambda_1 = 1 + 2\rho + 6\rho^2 = 5.34$$

$$\lambda_2 = \lambda_3 = 1 + 2\rho - 3\rho^2 = .93$$

$$\lambda_4 = \lambda_5 = \ldots = \lambda_9 = .3$$

Let $\underset{\sim}{Z}$ denote the (9x1) vector of standardized variables.

Let $\underset{\sim}{e}_1$, $\underset{\sim}{e}_2$, ..., $\underset{\sim}{e}_9$ denote the unit length eigenvectors corresponding to the eigenvalues λ_1, λ_2, ..., λ_9. The first principal component, $Y_1 = \underset{\sim}{e}_1'\underset{\sim}{Z}$, explains a proportion $\lambda_1/\text{trace} (\underset{\sim}{\rho}) = 5.34/9 = .539$ of the total variance of $\underset{\sim}{Z}$ (see Johnson and Wichern, 1982, chapter 8). The cumulative proportion of total variance explained by the first two principal components, $Y_1 = \underset{\sim}{e}_1'\underset{\sim}{Z}$ and $Y_2 = \underset{\sim}{e}_2'\underset{\sim}{Z}$, is $(5.34 + .93)/9 = .697$ and so on.

Consider the factor model (see Johnson and Wichern, 1982, chapter 9)

$$\underset{\substack{\sim \\ (9\times1)}}{Z} = \underset{\substack{\sim \\ (9\times k)}}{L} \quad \underset{\substack{\sim \\ (k\times1)}}{F} + \underset{\substack{\sim \\ (9\times1)}}{\varepsilon} \tag{19}$$

where $E(\underset{\sim}{F}) = 0$; $\text{Cov}(\underset{\sim}{F}) = I$; $E(\underset{\sim}{\varepsilon}) = 0$; $\text{Cov}(\underset{\sim}{\varepsilon}) = \Psi$ (diagonal); $E(\underset{\sim}{\varepsilon}\underset{\sim}{F}') = 0$ and $\underset{\sim\sim}{L}$ is a matrix of factor loadings. Let k=1. Representing the loadings by the scaled eigenvector $\underset{\sim}{e}_1$ (see Johnson and Wichern, 1982), we have $\underset{\sim}{L}' = \sqrt{\lambda_1}\,\underset{\sim}{e}_1' = \begin{bmatrix} .770, & .770, & \ldots, & .770 \end{bmatrix}$. The communalities are $h_i^2 = .593$, i = 1, 2, ..., 9 and hence the specific variances are $\Psi_i = .407$, i = 1, 2, ..., 9. The factor model gives

$$\text{Cov}(\underset{\sim}{Z}) = \underset{\sim}{L}\underset{\sim}{L}' + \Psi = \begin{bmatrix} 1 & .593 & . & . & . & .593 \\ & 1 & . & . & . & .593 \\ & & . & & & . \\ & & & . & & . \\ \text{(symmetric)} & & & & . & . \\ & & & & & 1 \end{bmatrix}$$

and the sum of squared errors in the "residual matrix," $\underset{\sim}{\rho} - (\underset{\sim}{LL'} + \Psi)$, is .779. The performance of k=2 and k=3 factor models can be analyzed in a similar fashion. Note for a k=2 factor model there is a choice for the loadings on the second factor since $\lambda_2 = \lambda_3$ and consequently, $L = \left[\sqrt{\lambda_1}\underset{\sim}{e}_1, \ \sqrt{\lambda_2}\underset{\sim}{e}_2\right]$ or $L = \left[\sqrt{\lambda_1}\underset{\sim}{e}_1, \ \sqrt{\lambda_3}\underset{\sim}{e}_3\right]$.

4. Correlation Matrix for Four Equally Correlated Groups

We consider four groups of stocks of the same sizes so m=n=p=q. Using the definitions in (1) and setting

$$\underset{(m \times m)}{C} = \begin{bmatrix} \rho^3 & \rho^3 & . & . & . & \rho^3 \\ \rho^3 & \rho^3 & . & . & . & \rho^3 \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ \rho^3 & \rho^3 & . & . & . & \rho^3 \end{bmatrix}$$

we consider the (4m×4m) correlation matrix

$$\underset{\sim}{\rho} = \begin{bmatrix} A & B & C & C \\ B & A & C & C \\ C & C & A & B \\ C & C & B & A \end{bmatrix} \tag{21}$$

where A, B and C are all of dimension (m×m). Thus we have identified two sets of groups (1,2), say, and (3,4).

If we regard the groups as stocks organized, for example, according to industries, then stocks within the same industries are equally correlated with the same correlation ρ for all industries. Stocks for industries 1 and 2,

respectively, are equally correlated with correlation ρ^2. A similar result holds for industries 3 and 4. Stocks in different industry groups, (1,2) and (3,4), respectively, are equally correlated with correlation ρ^3.

Given the structure of ρ in (21), it is easy to see that one of its eigen-values is $\lambda^{(1)} = 1 - \tilde{\rho}$ with multiplicity 4(m-1). The eigenvectors are of the same form as (4), (6) and (11) so we have

$$\lambda^{(1)} = 1 - \rho$$

$$\underset{\sim}{V}_1 = [\overbrace{1, -1, \ldots, 0}^{m}; \overbrace{0, \ldots, 0}^{m}; \overbrace{0, \ldots, 0}^{m}; \overbrace{0, \ldots, 0}^{m}]'$$

$$\vdots$$

$$\underset{\sim}{V}_{m-1} = [1, 1, \ldots, -(m-1); 0, \ldots, 0; 0, \ldots, 0; 0, \ldots, 0]'$$

$$\underset{\sim}{V}_m = [0, \ldots, 0; 1, -1, \ldots, 0; 0, \ldots, 0; 0, \ldots, 0]'$$

$$\vdots \qquad\qquad (22)$$

$$\underset{\sim}{V}_{2(m-1)} = [0, \ldots, 0; 1, 1, \ldots, -(m-1); 0, \ldots, 0; 0, \ldots, 0]'$$

$$\underset{\sim}{V}_{2m-1} = [0, \ldots, 0; 0, \ldots, 0; 1, -1, \ldots, 0; 0, \ldots, 0]'$$

$$\vdots$$

$$\underset{\sim}{V}_{3(m-1)} = [0, \ldots, 0; 0, \ldots, 0; 1, 1, \ldots, -(m-1); 0, \ldots, 0]'$$

$$\underset{\sim}{V}_{3m-2} = [0, \ldots, 0; 0, \ldots, 0; 0, \ldots, 0; 1, -1, \ldots, 0]'$$

$$\vdots$$

$$\underset{\sim}{V}_{4(m-1)} = [0, \ldots, 0; 0, \ldots, 0; 0, \ldots, 0; 1, 1, \ldots, -(m-1)]'$$

The remaining four eigenvectors must be of the form

$V = [a, \ldots, a; b, \ldots, b; c, \ldots, c; d, \ldots, d]'$ and must satisfy $\rho V = \lambda V$ or equivalently,

$$(1 + (m-1)\rho)a + m\rho^2 b + m\rho^3 c + m\rho^3 d = \lambda a$$

$$m\rho^2 a + (1 + (m-1)\rho)b + m\rho^3 c + m\rho^3 d = \lambda b$$

$$m\rho^3 a + m\rho^3 b + (1 + (m-1)\rho)c + m\rho^2 d = \lambda c \qquad (23)$$

$$m\rho^3 a + m\rho^3 b + m\rho^2 c + (1 + (m-1)\rho)d = \lambda d$$

Algebraic manipulations of the equations in (23), along the lines of the manipulations in section 3, yield the remaining distinct eigenvalues and associated, orthogonal, eigenvectors. These eigenvalues and (unnormalized) eigenvectors are:

$$\lambda^{(4)} = 1 + (m-1)\rho + m\rho^2 + 2m\rho^3$$

$$V_{4m} = [1, \ldots, 1; 1, \ldots, 1; 1, \ldots, 1; 1, \ldots, 1]'$$

$$\lambda^{(3)} = 1 + (m-1)\rho + m\rho^2 - 2m\rho^3$$

$$V_{4m-1} = [1, \ldots, 1; 1, \ldots, 1; -1, \ldots, -1; -1, \ldots, -1]' \qquad (24)$$

$$\lambda^{(2)} = 1 + (m-1)\rho - m\rho^2 \text{ (multiplicity 2)}$$

$$V_{4m-2} = [1, \ldots, 1; -1, \ldots, -1; 0, \ldots, 0; 0, \ldots, 0]'$$

$$V_{4m-3} = [0, \ldots, 0; 0, \ldots, 0; 1, \ldots, 1; -1, \ldots, -1]'$$

Regarding the eigenvectors as the coefficients in principal components constructed from standardized variables, we see that the principal component corresponding to the largest eigenvalue, $\lambda^{(4)}$, is a "sum" of the variables; the principal component corresponding to $\lambda^{(3)}$ represents a "contrast" between the groups

(1,2) and (3,4) and the principal components corresponding to $\lambda^{(2)}$ contrast collections 1 and 2 and collections 3 and 4 respectively.

Suppose the sample correlation matrix for the rates of return on m common stocks from each of, say, four different industries is of the form (21). Assume the factor model (19) with k=4. Taking the loadings matrix L to be $L = \left[\sqrt{\lambda_1}\underset{\sim}{e}_1, \sqrt{\lambda_2}\underset{\sim}{e}_2, \sqrt{\lambda_3}\underset{\sim}{e}_3, \sqrt{\lambda_4}\underset{\sim}{e}_4\right]$ where $\lambda_1 = \lambda^{(4)}$, $\underset{\sim}{e}_1 = \underset{\sim}{V}_{4m}/\|\underset{\sim}{V}_{4m}\|$; $\lambda_2 = \lambda^{(3)}$, $\underset{\sim}{e}_2 = \underset{\sim}{V}_{4m-1}/\|\underset{\sim}{V}_{4m-1}\|$; $\lambda_3 = \lambda_4 = \lambda^{(2)}$, $\underset{\sim}{e}_3 = \underset{\sim}{V}_{4m-2}/\|\underset{\sim}{V}_{4m-2}\|$ and $\underset{\sim}{e}_4 = \underset{\sim}{V}_{4m-3}/\|\underset{\sim}{V}_{4m-3}\|$; we might identify the first factor as a general market factor, the second factor as a bipolar factor contrasting industries (1,2) with industries (3,4). The third factor as a factor contrasting industries 1 and 2 and the fourth factor as a factor contrasting industries 3 and 4. Interpretations of this kind for stock prices have been suggested by King (1966). These interpretations are also consistent with the Arbitrage Pricing Theory (APT) of assets suggested by Ross (1976).

5. Patterned Correlation Matrices and Stock Price Behavior

The APT of Ross (1976) starts with an assumption on the stock return generating process. The return on the ith stock, X_i, is assumed to be generated by a k factor linear model of the form:

$$X_i = \mu_i + \ell_{i1} F_1 + \cdots + \ell_{ik} F_k + \varepsilon_i \tag{25}$$

$$i = 1, 2, \ldots, N$$

The first term, μ_i, is the expected return on the ith stock. The k factors, $F_1, \ldots, F_k$ are common to all stocks under consideration. The coefficients ℓ_{ij} (factor loadings) measure the reaction of stock i's returns to movements in the common factors F_j. The common factors represent the systematic risk components in the model. The error term, ε_i, represents the unsystematic risk component specific to the ith stock. Finally, it is assumed that N, the number of stocks

(assets), is much greater than k, the number of underlying common factors. If the expected returns are subtracted from the returns and the risk components are rescaled, the model in (25) is analogous to the linear factor model (19).

Beginning with the linear factor model for returns, Ross develops an equilibrium "pricing" relationship for stocks (assets) which must hold in the absence of riskless arbitrage opportunities. The accompanying APT has achieved a significant level of acceptance in the finance academic community. It has been tested, empirically, with varying degrees of support. In particular, Roll and Ross (1980) claim to find support for the APT in their empirical investigation and suggest the number of common factors is around four or five. It is the latter result, along with our frequently observed "regular" correlations between stock returns, that motivated our investigations in the previous sections.

As a small "real world" example, Johnson and Wichern (1982, chapter 8) report the correlation matrix, R, constructed from one hundred daily rate of returns for each of five stocks. Three of the stocks are chemicals (Allied Chemical, DuPont and Union Carbide) and two of the stocks are oils (Exxon and Texaco). The correlation matrix and its eigenvalues and normalized eigenvectors are given below.

$$
R = \left[\begin{array}{ccc|cc}
\overbrace{1.0 \quad .577 \quad .509}^{\text{chemicals}} & .387 & .462 \\
1.0 \quad .599 & .389 & .322 \\
1.0 & .436 & .426 \\
\hline
\text{(symmetric)} & 1.0 & .523 \\
& & 1.0
\end{array}\right] \Bigg\} \quad \text{oils}
$$

$\lambda_1 = 2.857, \quad e_1 = [.464, .457, .470, .421, .421]'$

$\lambda_2 = .809, \quad e_2 = [.240, .509, .260, -.526, -.582]'$

$\lambda_3 = .540, \quad e_3 = [-.621, .178, .335, .541, -.435]'$

$\lambda_4 = .452, \quad e_4 = [.387, .206, -.662, .472, -.382]'$

$\lambda_5 = .343, \quad e_5 = [-.451, .676, -.400, -.176, .385]'$

It is clear we have two distinct groups of stocks with roughly the same within group correlation ($\bar{r}$ = .55) and (very) roughly the same between group correlation ($\bar{r}$ = .40). The first two sample principal components, $Y_1 = \underset{\sim}{e}_1'\underset{\sim}{Z}$ and $Y_2 = \underset{\sim}{e}_2'\underset{\sim}{Z}$ explain a proportion (2.857 + .809)/5 = .73 of the total variance of the standardized returns. Clearly, the first component represents a "sum" of the five stocks and the second component represents a "contrast" between the chemicals and oils.

Assuming a k=2 factor model of the form (19) for the stock returns and estimating the loading matrix, L, with $\hat{L} = \left[\sqrt{\lambda_1}\underset{\sim}{e}_1, \sqrt{\lambda_2}\underset{\sim}{e}_2\right]$ we might identify the first factor as a "market factor" and the second factor as an "industry" factor (see, also, Johnson and Wichern, 1982, chapter 9). All stocks have large and nearly equal loadings on the first factor. The second factor is a bipolar factor. The chemical stocks have positive loadings and the oil stocks have negative loadings on this factor.

Comparing the correlation matrix, R, above to the correlation matrix $\underset{\sim}{\rho}$ discussed in the numerical example of section 2, we see that there is some (descriptive) evidence that the correlation patterns among stocks are somewhat regular. Specifically, if we consider two groups of stocks with m=3, n=2 and a population correlation matrix of the form (2) with ρ = .6 (the case considered in the numerical example of section 2), the first two (important) eigenvalues and normalized eigenvectors of $\underset{\sim}{\rho}$ are

$$\lambda_1 = 2.832; \ \underset{\sim}{e}_1 = \left[.469, .469, .469, .412, .412\right]'$$

$$\lambda_2 = .969; \ \underset{\sim}{e}_2 = \left[.336, .336, .336, -.575, -.575\right]'$$

These are very similar to the first two eigenvalues and eigenvectors of the sample correlation matrix R. Moreover, the three smallest eigenvalues of R are

roughly equal ($\bar{\lambda} = .445$ for λ_3, λ_4 and λ_5) and consistent with the final three eigenvalues of $\underset{\sim}{\rho}$ ($\lambda_3 = \lambda_4 = \lambda_5 = .400$).

We have provided some evidence for the linear factor model for stock returns assumed by Ross. In addition we have suggested that the measures of systematic risk (the factor loadings) may be more regular than initially proposed.

6. Summary

We have derived general expressions for the eigenvalues and eigenvectors of certain patterned correlation matrices. The particular patterns examined were suggested by observed correlations among common stock returns. We have discussed our results in the context of a principal components or factor analysis of stock returns. The limited evidence presented here suggests the linear factor model for stock returns suggested by Ross as part of the Arbitrage Pricing Theory of assets is not unreasonable. Moreover, the factor loadings (measures of the response of the returns to the systematic risk components of the model) may be equal for distinct groups of stocks. This implies more regularity in the generating mechanism for stock returns than previously imagined. However, more work is needed before this assertion is confirmed.

Finally, we note that the derivations for the matrix $\underset{\sim}{\rho}$ (see (21)) presented in section 4 can be extended to the case of several matrices B and C appearing in each row. However, the numerical calculations involved become rather cumbersome.

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General forms for the eigenvalues and eigenvectors of certain patterned correlation matrices are obtained. The results are discussed in the context of a factor analysis of common stock rates of return.

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